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Relation between Polynomials Orthogonal on the Unit Circle with Respect to Different Weights*

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Formulas of Christoffel and Uvarov for changing the weight of real orthogonal polynomials by polynomial multiplication and division are extended to polynomials orthogonal on the unit circle.

1. INTRODUCTION

Let $\{\phi_n(z)\}$ be the orthonormal polynomials associated with the weight $dv(\theta)$ on the unit circle, $z = e^{i\theta}$. We wish to find a representation of the polynomials orthogonal with respect to a new weight, $g(\theta)\, dv(\theta)$, where $g(\theta)$ is either a non-negative trigonometric polynomial or the reciprocal of a positive trigonometric polynomial. We then find, up to multiplicative constants, the polynomials associated with the quotient of a non-negative trigonometric polynomial over a positive trigonometric polynomial. A result of L. Fejér and F. Riesz [6, pp. 20–22] states that every non-negative trigonometric polynomial can be written in the form $z^{-m}G_{2m}(z)$. Thus we consider $G_{2m}(z)$ such that

$$z^{-m}G_{2m}(z) = |G_{2m}(z)|, \qquad z = e^{i\theta}$$

with $G_{2m}(z)$ of precise degree $2m$.

Our first result is an extension of a formula of Christoffel and deals with the case when $g(\theta)$ is a non-negative trigonometric polynomial. It is stated as Theorem 1 below. The main idea behind the proof of Theorem 1 is similar to that found in Szegö [15, pp. 29–31] of "a formula of Christoffel" due to Christoffel. This is to be expected, for what follows is a complex analogue of what happens in the real case.

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V. B. Uvarov [16, 17] generalized this formula of Christoffel to include rational weights. As we shall see his inversion also has an analogue on the circle. In Section 3 we derive a determinant representation for the polynomials orthogonal with respect to $g(\theta)\,dv(\theta)$ when $g(\theta)$ is a quotient of two trigonometric polynomials.

THEOREM 1. *Let $\{\phi_n(z)\}$ be the orthonormal polynomials associated with the distribution $dv(\theta)$ on $z=e^{i\theta}$. Let $G_{2m}(z)$ be a polynomial of precise degree $2m$ such that*

$$z^{-m}G_{2m}(z)=|G_{2m}(z)|, \qquad |z|=1.$$

Finally, let $\{\psi_n(z)\}$ be given by

$$G_{2m}(z)\psi_n(z)$$
$$=\det\begin{pmatrix} \phi^*(z) & z\phi^*(z) & \cdots & z^{m-1}\phi^*(z) & \phi(z) & z\phi(z) & \cdots & z^m\phi(z) \\ \phi^*(\alpha_1) & \alpha_1\phi^*(\alpha_1) & \cdots & \alpha_1^{m-1}\phi^*(\alpha_1) & \phi(\alpha_1) & \alpha_1\phi(\alpha_1) & \cdots & \alpha_1^m\phi(\alpha_1) \\ \phi^*(\alpha_2) & \alpha_2\phi^*(\alpha_2) & \cdots & \alpha_2^{m-1}\phi^*(\alpha_2) & \phi(\alpha_2) & \alpha_2\phi(\alpha_2) & \cdots & \alpha_2^m\phi(\alpha_2) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \phi^*(\alpha_{2m}) & \alpha_{2m}\phi^*(\alpha_{2m}) & \cdots & \alpha_{2m}^{m-1}\phi^*(\alpha_{2m}) & \phi(\alpha_{2m}) & \alpha_{2m}\phi(\alpha_{2m}) & \cdots & \alpha_{2m}^m\phi(\alpha_{2m}) \end{pmatrix}, \quad (1)$$

where the zeros of $G_{2m}(z)$ are $\{\alpha_1, \alpha_2, ..., \alpha_{2m}\}$ and $\phi(z)$ denotes $\phi_{n+m}(z)$.

For zeros of multiplicity h, $h>1$, we replace the corresponding rows in the determinant by the derivatives of order 0, 1, 2, ..., $h-1$ *of the polynomials in the first row, evaluated at that zero. (As usual, $\rho_r^*(z)=z^r\bar{\rho}_r(z^{-1})$, for $\rho_r(z)$ a polynomial of degree r.)*

Then $\{\psi_n(z)\}$ are orthogonal polynomials associated with the distribution $|G_{2m}(z)|\,dv(\theta)$ on the unit circle, $z=e^{i\theta}$.

Our proof of Theorem 1 requires the following lemmas.

LEMMA 1. *Each of the polynomials in the first row, when divided by z^m, is orthogonal to an arbitrary polynomial of degree at most $n-1$.*

LEMMA 2. *The above determinant is a polynomial of precise degree $2m+n$.*

The above theorem and lemmas will be proved in Section 2. In Section 3 we treat the case of multiplication of the weight by the reciprocal of a positive trigonometric polynomial; see Theorem 2.

Let $p(x)$ be a polynomial positive on $[-1, 1]$ and of precise degree m. Bernstein and Szegö studied the class of polynomials orthogonal on $[-1, 1]$ with repsect to the weight functions w_1, w_2, w_3,

$$w_1(x)=(1-x^2)^{-1/2}/p(x), \qquad w_2(x)=(1-x^2)w_1(x),$$
$$w_3(x)=(1-x)w_1(x).$$

Explicit formulas for the orthogonal polynomials in all three cases are in Szegö [15, Sect. 2.6]. The orghogonal polynomials associated with $w_1(x)$ played an important role in Szegö's proof of his equiconvergence theorem [15, Theorem 13.1.2]. Bernstein used the orthogonal polynomials in all three cases in connection with his asymptotic formula [15, Theorem 12.1.4] which describes the large n behavior of general polynomials orthogonal with respect to a weight function satisfying certain local conditions.

Using the Szegö mapping [15, Sect. 11.5] between polynomials orthogonal on the interval $[-1, 1]$ and polynomials orthogonal on the unit circle one can find explicit formulas for the orthogonal polynomials on the circle associated with the weights $1/p(\cos\theta)$, $\sin^2\theta/p(\cos\theta)$, $(1-\cos\theta)/p(\cos\theta)$. These are special cases of our Theorem 2 when $v(\theta)=\theta$ and $g(\theta)$ is an even function of θ.

In Section 4 we state the result for the quotient of a non-negative trigonometric polynomial over a positive one. This result is Theorem 3. In Section 4 we also apply Theorem 3 to the case $v(\theta)=\theta$ and record an explicit representation for the orthogonal polynomials associated with weight functions $|G_{2m}(e^{i\theta})/H_{2k}(e^{i\theta})|\,d\theta$ if $e^{-im\theta}G_{2m}(e^{i\theta})$ is a non-negative trigonometric polynomial and $e^{-ik\theta}H_{2k}(e^{i\theta})$ is a positive one. We also show how Theorem 4 through the aforementioned explicit formulas gives the strong asymptotics for polynomials orthogonal on the unit circle with respect to quotients of trigonometric polynomials. This latter result may be useful in investigating the strong asymptotics of orthogonal polynomials by approximating their weight functions by quotients of trigonometric polynomials; see Grenander and Szegö [6] and Máté, Nevai, and Totik [11, 12].

In Section 5 we show how the above Theorem 1 gives an alternative way of deriving the explicit formula for the polynomials studied by Szegö and generalized by Askey and Hahn.

2. Proofs

This section contains proofs of Lemmas 1 and 2 and Theorem 1. We start with Lemma 1.

Proof of Lemma 1. We only have two types of polynomials in the first row of the determinant. We consider each separately. Let $\rho_{n-1}(z)$ be any polynomial of degree at most $n-1$. Then, for the polynomials $z^l\phi_{n+m}(z)$, where $l=0, 1, 2, ..., m$, we have

$$\int_{-\pi}^{\pi} \frac{z^l\phi_{n+m}(z)}{z^m}\,\overline{\rho_{n-1}(z)}\,dv(\theta)=\int_{-\pi}^{\pi} \phi_{n+m}(z)\,\overline{z^{m-l}\rho_{n-1}(z)}\,dv(\theta)=0.$$

For the polynomials $z^l\phi^*_{n+m}(z)$ we have

$$\int_{-\pi}^{\pi} \overline{\left(\frac{z^l\phi^*_{n+m}(z)}{z^m}\right)}\, \rho_{n-1}(z)\, dv(\theta) = \int_{-\pi}^{\pi} z^{m-l}\,\overline{\phi^*_{n+m}(z)}\, \rho_{n-1}(z)\, dv(\theta)$$

$$= \int_{-\pi}^{\pi} z^{m-l}\,\overline{z^{n+m}\bar{\phi}_{n+m}(1/z)}\,\rho_{n-1}(z)\, dv(\theta)$$

$$= \int_{-\pi}^{\pi} \phi_{n+m}(z) z^{-n-l} \rho_{n-1}(z) dv(\theta)$$

$$= \int_{-\pi}^{\pi} \phi_{n+m}(z)\overline{z^{n+l}\bar{\rho}_{n-1}(1/z)}\, dv(\theta)$$

$$= \int_{-\pi}^{\pi} \phi_{n+m}(z)\overline{z^{l+1}\rho^*_{n-1}(z)}\, dv(\theta) = 0$$

for $l = 0, 1, 2, ..., m-1$. This completes the proof of Lemma 1.

Proof of Lemma 2. Assume the coefficient of $z^m\phi_{n+m}(z)$ is zero; i.e., the determinant we get from crossing out the first row and last column of our original matrix is zero. Then there exist constants, not all zero, $\lambda_0, \lambda_1, ..., \lambda_{m-1}$ and $\gamma_0, \gamma_1, ..., \gamma_{m-1}$, such that the polynomial $g(z)$ defined by

$$g(z) := (\lambda_0 + \lambda_1 z + \cdots + \lambda_{m-1} z^{m-1})\, \phi_{n+m}(z) \\ + (\gamma_0 + \gamma_1 z + \cdots + \gamma_{m-1} z^{m-1}) \phi^*_{n+m}(z)$$

vanishes for $z = \alpha_1, \alpha_2, ..., \alpha_{2m}$. This shows that $g(z)$ has the form $g(z) = G_{2m}(z)\rho_{n-1}(z)$ for some $\rho_{n-1}(z)$. We know that $g(z)$ is not identically zero as the zeros of $\phi(z)$ lie in $|z| < 1$ and the zeros of $\phi^*(z)$ lie in $|z| > 1$. From Lemma 1 we know $g(z)/z^m$ is orthogonal to any polynomial of degree less than n. Thus,

$$0 = \int_{-\pi}^{\pi} \frac{g(z)}{z^m}\,\overline{\rho_{n-1}(z)}\, dv(\theta) = \int_{-\pi}^{\pi} \frac{G_{2m}(z)\rho_{n-1}(z)}{z^m}\,\overline{\rho_{n-1}(z)}\, dv(\theta)$$

$$= \int_{-\pi}^{\pi} |\rho_{n-1}(z)|^2\, |G_{2m}(z)|\, dv(\theta)$$

which implies $\rho_{n-1}(z) \equiv 0$ and consequently $g(z) \equiv 0$.

Proof of Theorem 1. From Lemma 2 and the form of the determinant in (1) each $\psi_n(z)$ is a polynomial of degree n. From Lemma 1 we see that for any $\rho_{n-1}(z)$

$$\int_{-\pi}^{\pi} \frac{G_{2m}(z)\psi_n(z)}{z^m} \overline{\rho_{n-1}(z)}\, dv(\theta) = 0;$$

that is,

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\rho_{n-1}(z)}\, |G_{2m}(z)|\, dv(\theta) = 0.$$

Thus the polynomials $\{\psi_n(z)\}$ are constant multiples of the polynomials orthonormal with respect to $|G_{2m}(z)|\, dv(\theta)$.

3. Rational Weights

We can combine the above Theorem 1 with a result of Szegö to obtain a determinant representation for the polynomials orthogonal with respect to a positive trigonometric rational function.

Specifically, assume $g_1(\theta)$ and $g_2(\theta)$ are positive trigonometric polynomials. From Szegö [15, Theorem 11.2] we get most of the polynomials associated with the weight $(1/g_2(\theta))\, d\theta$,

$$\phi_n(z) = z^{n-r} h^*(z), \qquad \text{for} \quad n = r, r+1, ...,$$

where $g_2(\theta) = |h(z)|^2$, $z = e^{i\theta}$, $g_2(\theta)$ and $h(z)$ both have degree r, $h(z) \neq 0$ in $|z| < 1$, and $h(0) > 0$.

If we substitute these into our determinant representation, using $g_1(\theta) = z^{-m} G_{2m}(z)$ we can get a representation for the orthogonal polynomials associated with $(g_1(\theta)/g_2(\theta))\, d\theta$. (Note that if the degree of $g_1(\theta)$ is $\geqslant$ the degree of $g_2(\theta)$ then we are missing none of the orthogonal polynomials; that is, our representation is valid for $\psi_0(z), \psi_1(z), \psi_2(z), ...$).

More interesting than this, V. B. Uvarov [16, 17] generalized the original theorem of Christoffel to include multiplication by rational weights and, in particular, division of the weight function by a positive polynomial.

Once again we start with the Fejér–Riesz result and write

$$\frac{1}{g(\theta)} = \frac{z^m}{G_{2m}(z)}, \qquad |z| = 1,$$

where $g(\theta)$ is now a positive trigonometric polynomial and, of course, $G_{2m}(z)$ has no zeros on $|z| = 1$.

THEOREM 2. *Let $\{\phi_n(z)\}$ be the orthonormal polynomials associated with the distribution $dv(\theta)$ on $z=e^{i\theta}$ and let $G_{2m}(z)$ be a polynomial of precise degree $2m$ such that*

$$z^{-m}G_{2m}(z)=|G_{2m}(z)|>0, \qquad z=e^{i\theta}.$$

Finally, define a system of polynomials $\{\psi_n(z)\}$, $n=2m, 2m+1, ...,$ by

$$\psi_n(z)=\det\begin{pmatrix} \phi^*(z) & z\phi^*(z) & \cdots & z^{m-1}\phi^*(z) & \phi(z) & z\phi(z) & \cdots & z^m\phi(z) \\ L_{\beta_1}(\phi^*) & L_{\beta_1}(z\phi^*) & \cdots & L_{\beta_1}(z^{m-1}\phi^*) & L_{\beta_1}(\phi) & L_{\beta_1}(z\phi) & \cdots & L_{\beta_1}(z^m\phi) \\ L_{\beta_2}(\phi^*) & L_{\beta_2}(z\phi^*) & \cdots & L_{\beta_2}(z^{m-1}\phi^*) & L_{\beta_2}(\phi) & L_{\beta_2}(z\phi) & \cdots & L_{\beta_2}(z^m\phi) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ L_{\beta_{2m}}(\phi^*) & L_{\beta_{2m}}(z\phi^*) & \cdots & L_{\beta_{2m}}(z^{m-1}\phi^*) & L_{\beta_{2m}}(\phi) & L_{\beta_{2m}}(z\phi) & \cdots & L_{\beta_{2m}}(z^m\phi) \end{pmatrix}, \qquad (2)$$

where the zeros of $G_{2m}(z)$ are $\{\beta_1, \beta_2, ..., \beta_{2m}\}$, $\phi(z)$ denotes $\phi_{n-m}(z)$, and where we define

$$L_\beta(p):=\int_{-\pi}^{\pi} p(\xi)\overline{\left(\frac{\xi^m}{\xi-\beta}\right)}\,dv(\theta), \qquad \xi=e^{i\theta}.$$

For zeros of multiplicity h, $h>1$, we replace the corresponding rows in the determinant in (2) *by*

$$L_\beta^k(p):=\int_{-\pi}^{\pi} p(\xi)\overline{\left(\frac{\xi^m}{(\xi-\beta)^k}\right)}\,dv(\theta), \qquad \xi=e^{i\theta},$$

$k=1, 2, ..., h$ acting on the first row.

Under the above assumptions $\{\psi_n(z)\}$ are the orthonormal polynomials associated with the distribution $(1/|G_{2m}(z)|)\,dv(\theta)$ on the unit circle, $z=e^{i\theta}$, up to multiplicative constants, for $n\geqslant 2m$.

Proof of Theorem 2. Assume for the moment that the zeros of $G_{2m}(z)$ are pairwise distinct.

Now, if $k\geqslant 2m$ and $\rho_k(z)$ is of precise degree k we have

$$\rho_k(z)=G_{2m}(z)q(z)+r(z)$$

with the degree of $r(z)$ less than $2m$. Thus define

$$q_{k-2m}(z)=\frac{\rho_k(z)}{G_{2m}(z)}-\frac{r(z)}{G_{2m}(z)},$$

where in case $k<2m$ we set $r(z)\equiv\rho_k(z)$ and $q_{k-2m}(z)\equiv 0$. In either case $q_{k-2m}(z)$ has degree at most $k-2m$.

Now we decompose $r(z)/G_{2m}(z)$ via partial fractions, i.e.,

$$\frac{r(z)}{G_{2m}(z)}=\sum_{i=1}^{2m}\frac{A_i(\rho_k)}{z-\beta_i},$$

where the $\{A_i(\rho_k)\}$ are constants depending on ρ_k. Assuming $k \leqslant n-1$ we have for every

$$\gamma(z)\in \operatorname{Span}\{\phi(z), \phi^*(z), z\phi(z), z\phi^*(z), ..., z^{m-1}\phi(z), z^{m-1}\phi^*(z), z^m\phi(z)\},$$

where ϕ denotes ϕ_{n-m}, that

$$\int_{-\pi}^{\pi}\gamma(z)\overline{z^m q_{k-2m}(z)}\,dv(\theta)=0$$

and thus

$$\int_{-\pi}^{\pi}\gamma(z)\overline{\rho_k(z)}\frac{1}{|G_{2m}(z)}\,dv(\theta)=\sum_{i=1}^{2m}\left[\overline{A_i(\rho_k)}\int_{-\pi}^{\pi}\gamma(z)\overline{\left(\frac{z^m}{z-\beta_i}\right)}\,dv(\theta)\right]$$

for $k \leqslant n-1$.

Hence if we let $\psi_n(z)$ be defined as in the above Theorem 2 we get

$$\int_{-\pi}^{\pi}\psi_n(z)\overline{\rho_k(z)}\frac{1}{|G_{2m}(z)|}\,d(\theta)=0, \qquad k\leqslant n-1$$

by linearity as under integration the first row in the determinant will be a linear combination of the lower rows. (If $G_{2m}(z)$ has multiple zeros we simply change the form of the partial fraction decomposition.) However, we still must show that $\psi_n(z)$ is of precise degree n. For that we will require $n \geqslant 2m$. Thus we are missing the first $2m$ polynomials in our representation.

Assume the coefficient of $z^m\phi_{n-m}(z)$ is zero; i.e., the determinant we get from crossing out the first row and last column of our matrix is zero. Then there exist constants $\lambda_0, \lambda_1, ..., \lambda_{m-1}$ and $\mu_0, \mu_1, ..., \mu_{m-1}$, not all zero, such that if we let $\gamma(z)$ be defined by

$$\begin{aligned}\gamma(z) &:= (\lambda_0+\lambda_1+\cdots+\lambda_{m-1}z^{m-1})\phi_{n-m}(z)\\ &\quad+(\mu_0+\mu_1 z+\cdots+\mu_{m-1}z^{m-1})\phi^*_{n-m}(z)\end{aligned}$$

we have $L_{\beta_i}(\gamma)=0$ for every i.

This means

$$\int_{-\pi}^{\pi}\gamma(z)\overline{\rho_k(z)}\frac{1}{|G_{2m}(z)|}\,dv(\theta)=0$$

for every polynomial $\rho_k(z)$ of degree $k \leqslant n-1$ and, in particular, for $\gamma(z)$ as well.

Thus

$$\int_{-\pi}^{\pi} |\gamma(z)|^2 \frac{1}{|G_{2m}(z)|} \, dv(\theta) = 0,$$

which implies that $\gamma(z) \equiv 0$. However, if $n \geqslant 2m$ then $\gamma(z)$ cannot be identically zero. Thus the polynomials $\{\psi(z)\}$ are constant multiples of the polynomials orthonormal with respect to $(1/|G_{2m}(z)|)\, dv(\theta)$.

4. General Rational Weights and Applications

We may combine the proofs of Theorems 1 and 2 to produce the following theorem.

Theorem 3. *Let $\{\phi_n(z)\}$ be the orthonormal polynomials associated with the distribution $dv(\theta)$ on the unit circle, $z = e^{i\theta}$, and let $G_{2m}(z)$ and $H_{2k}(z)$ be polynomials of precise degrees $2m$ and $2k$, respectively, such that*

$$z^{-m} G_{2m}(z) = |G_{2m}(z)|, \qquad z^{-k} H_{2k}(z) = |H_{2k}(z)| > 0, \qquad |z| = 1.$$

Assume the zeros of $G_{2m}(z)$ are $\{\alpha_1, \alpha_2, ..., \alpha_{2m}\}$ and the zeros of $H_{2k}(z)$ are $\{\beta_1, \beta_2, ..., \beta_{2k}\}$. Let $\phi(z)$ denote $\phi_{n+m-k}(z)$ and $s = m+k$. For $n \geqslant 2k$ define $\psi_n(z)$ by

$$G_{2m}(z)\psi_n(z)$$

$$= \det \begin{bmatrix} \phi^*(z) & z\phi^*(z) & \cdots & z^{s-1}\phi^*(z) & \phi(z) & z\phi(z) & \cdots & z^s\phi(z) \\ \phi^*(\alpha_1) & \alpha_1\phi^*(\alpha_1) & \cdots & \alpha_1^{s-1}\phi^*(\alpha_1) & \phi(\alpha_1) & \alpha_1\phi(\alpha_1) & \cdots & \alpha_1^s\phi(\alpha_1) \\ \phi^*(\alpha_2) & \alpha_2\phi^*(\alpha_2) & \cdots & \alpha_2^{s-1}\phi^*(\alpha_2) & \phi(\alpha^2) & \alpha_2\phi(\alpha_2) & \cdots & \alpha_2^s\phi(\alpha_2) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \phi^*(\alpha_{2m}) & \alpha_{2m}\phi^*(\alpha_{2m}) & \cdots & \alpha_{2m}^{s-1}\phi^*(\alpha_{2m}) & \phi(\alpha_{2m}) & \alpha_{2m}\phi(\alpha_{2m}) & \cdots & \alpha_{2m}^s\phi(\alpha_{2m}) \\ L_{\beta_1}(\phi^*) & L_{\beta_1}(z\phi^*) & \cdots & L_{\beta_1}(z^{s-1}\phi^*) & L_{\beta_1}(\phi) & L_{\beta_1}(z\phi) & \cdots & L_{\beta_1}(z^s\phi) \\ L_{\beta_2}(\phi^*) & L_{\beta_2}(z\phi^*) & \cdots & L_{\beta_2}(z^{s-1}\phi^*) & L_{\beta_2}(\phi) & L_{\beta_2}(z\phi) & \cdots & L_{\beta_2}(z^s\phi) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ L_{\beta_{2k}}(\phi^*) & L_{\beta_{2k}}(z\phi^*) & \cdots & L_{\beta_{2k}}(z^{s-1}\phi^*) & L_{\beta_{2k}}(z\phi) & L_{\beta_{2k}}(z\phi) & \cdots & L_{\beta_{2k}}(z^s\phi) \end{bmatrix}, \tag{3}$$

where we define

$$L_\beta(p) := \int_{-\pi}^{\pi} p(\xi)\overline{(\xi^s/(\xi-\beta))}\, dv(\theta), \qquad \xi = e^{i\theta}.$$

For zeros of $H_{2k}(z)$ of multiplicity h, $h>1$, we replace the corresponding rows in the determinant by

$$L^r_\beta(p) := \int_{-\pi}^{\pi} p(\xi)\overline{(\xi^s/(\xi-\beta)^r)}\,dv(\theta), \qquad \xi = e^{i\theta},$$

$r = 1, 2, ..., h$ acting on the first row.

For zeros of $G_{2m}(z)$ of multiplicity h, $h>1$, we replace the corresponding rows in the determinant by the derivatives of order $0, 1, 2, ..., h-1$ of the polynomials in the first row, evaluated at that zero. (As usual, $\rho_r^*(z) = z^r\bar{\rho}_r(z^{-1})$, for $\rho_r(z)$ a polynomial of degree r.)

Then $\{\psi_n(z)\}$ are constant multiples of the polynomials orthonormal with respect to $|G_{2m}(z)/H_{2k}(z)|\,dv(\theta)$ on the unit circle, $z=e^{i\theta}$.

Applications

The case when $dv(\theta) = d\theta$ is fairly straightforward. For then we have $\phi_n(z) = z^n$, $\phi^*(z) = 1$, and

$$L_\beta(z^n) = \begin{cases} -2\pi(\beta^*)^{n-s+1}, & |\beta|>1, \quad n \geqslant s \\ 0, & |\beta|<1, \quad n \geqslant s \\ 2\pi(\beta^*)^{n-s+1}, & |\beta|<1, \quad n<s \\ 0, & |\beta|>1, \quad n<s, \end{cases}$$

where $\beta^* = 1/\bar{\beta}$, $|\beta| \neq 1$. Thus the determinants in Theorems 1, 2, and 3 simplify greatly.

Application 1. We may see how our version of Uvarov's generalization matches with Szegö's theorem for the case where the weight to be obtained is $(1/|H_{2m}(z)|)\,d\theta$. We obtain, for some suitable constant λ_n, that

$$\psi_{n+m}(z)$$

$$= \lambda_n \det \begin{bmatrix} 1 & z & z^2 & \cdots & z^{m-1} & z^n & z^{n+1} & z^{n+2} & \cdots & z^{n+m} \\ 1 & \beta_1^* & \beta_1^{*2} & \cdots & \beta_1^{*m-1} & 0 & 0 & 0 & \cdots & 0 \\ 1 & \beta_2^* & \beta_2^{*2} & \cdots & \beta_2^{*m-1} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \beta_m^* & \beta_m^{*2} & \cdots & \beta_m^{*m-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^m \\ 0 & 0 & 0 & \cdots & 0 & 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^m \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \beta_m & \beta_m^2 & \cdots & \beta_m^m \end{bmatrix}$$

where $|\beta_i^*| > 1$ and $|\beta_i| < 1$ are the zeros of $H_{2m}(z)$. Using Laplace's

Theorem we may express the determinant as a product of Vandermonde determinants and thus get

$$\psi_{n+m}(z)=c_n z^n \prod_{i=1}^{m} (z-\beta_i)$$

for some suitable constant c_n.

Application 2. Next we consider the rational trigonometric weight $|G_{2m}(z)/H_{2k}(z)|\, d\theta$. Moreover, assume the zeros of $G_{2m}(z)$ and those of $H_{2k}(z)$ are pairwise distinct. Let $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be the zeros of $G_{2m}(z)$ in $|z|<1$ and $\{\alpha_1^*, \alpha_2^*, ..., \alpha_m^*\}$ its zeros in $|z|>1$. Let $\{\beta_1, \beta_2, ..., \beta_k\}$ be the zeros of $H_{2k}(z)$ in $|z|<1$ and $\{\beta_1^*, \beta_2^*, ..., \beta_k^*\}$ its zeros in $|z|>1$. Finally, let $s=m+k$, $r=n+m-k$, and assume $n \geqslant 2k$. In this case Theorem 3 gives, for a suitable constant λ_n, that

$$G_{2m}(z)\psi_n(z)$$

$$=\lambda_n \det \begin{bmatrix} 1 & z & z^2 & \cdots & z^{s-1} & z^r & z^{r+1} & z^{r+2} & \cdots & z^{r+s} \\ 1 & \alpha_1^* & \alpha_1^{*2} & \cdots & \alpha_1^{*s-1} & \alpha_1^{*r} & \alpha_1^{*r+1} & \alpha_1^{*r+2} & \cdots & \alpha_1^{*r+s} \\ 1 & \alpha_2^* & \alpha_2^{*2} & \cdots & \alpha_2^{*s-1} & \alpha_2^{*r} & \alpha_2^{*r+1} & \alpha_2^{*r+2} & \cdots & \alpha_2^{*r+s} \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_m^* & \alpha_m^{*2} & \cdots & \alpha_m^{*s-1} & \alpha_m^{*r} & \alpha_m^{*r+1} & \alpha_m^{*r+2} & \cdots & \alpha_m^{*r+s} \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{s-1} & \alpha_1^r & \alpha_1^{r+1} & \alpha_1^{r+2} & \cdots & \alpha_1^{r+s} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{s-1} & \alpha_2^r & \alpha_2^{r+1} & \alpha_2^{r+2} & \cdots & \alpha_2^{r+s} \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \cdots & \alpha_m^{s-1} & \alpha_m^r & \alpha_m^{r+1} & \alpha_m^{r+2} & \cdots & \alpha_m^{r+s} \\ 1 & \beta_1^* & \beta_1^{*2} & \cdots & \beta_1^{*s-1} & 0 & 0 & 0 & \cdots & 0 \\ 1 & \beta_2^* & \beta_2^{*2} & \cdots & \beta_2^{*s-1} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \beta_k^* & \beta_k^{*2} & \cdots & \beta_k^{*s-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^s \\ 0 & 0 & 0 & \cdots & 0 & 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^s \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \beta_k & \beta_k^2 & \cdots & \beta_k^s \end{bmatrix}.$$

Now we will be more precise and make both sides of the last equation monic. Let $\Psi_n(z)$ denote the corresponding monic orthogonal polynomials and set

$$c=(-1)^m \prod_{i=1}^{m} \alpha_i^*$$

so that $cG_{2m}(z)$ will be monic as well. Finally, we manipulate the determinant and write

$$cG_{2m}(z)\,\Psi_n(z)=\frac{\det(A_n(z))}{\det(B_n)},$$

where

$$A_n(z)=\begin{bmatrix}
1 & z & z^2 & \cdots & z^{s-1} & z^r & z^{r+1} & z^{r+2} & \cdots & z^{r+s}\\
1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{s-1} & \alpha_1^r & \alpha_1^{r+1} & \alpha_1^{r+2} & \cdots & \alpha_1^{r+s}\\
1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{s-1} & \alpha_2^r & \alpha_2^{r-1} & \alpha_2^{r+2} & \cdots & \alpha_2^{r+s}\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots\\
1 & \alpha_m & \alpha_m^2 & \cdots & \alpha_m^{s-1} & \alpha_m^r & \alpha_m^{r+1} & \alpha_m^{r+2} & \cdots & \alpha_m^{r+s}\\
1 & \beta_1^* & \beta_1^{*2} & \cdots & \beta_1^{*s-1} & 0 & 0 & 0 & \cdots & 0\\
1 & \beta_2^* & \beta_2^{*2} & \cdots & \beta_2^{*s-1} & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots\\
1 & \beta_k^* & \beta_k^{*2} & \cdots & \beta_k^{*s-1} & 0 & 0 & 0 & \cdots & 0\\
(\bar{\alpha}_1)^r & (\bar{\alpha}_1)^{r-1} & (\bar{\alpha}_1)^{r-2} & \cdots & (\bar{\alpha}_1)^{r-s+1} & 1 & \alpha_1^* & \alpha_1^{*2} & \cdots & \alpha_1^{*s}\\
(\bar{\alpha}_2)^r & (\bar{\alpha}_2)^{r-1} & (\bar{\alpha}_2)^{r-2} & \cdots & (\bar{\alpha}_2)^{r-s+1} & 1 & \alpha_2^* & \alpha_2^{*2} & \cdots & \alpha_2^{*s}\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots\\
(\bar{\alpha}_m)^r & (\bar{\alpha}_m)^{r-1} & (\bar{\alpha}_m)^{r-2} & \cdots & (\bar{\alpha}_m)^{r-s+1} & 1 & \alpha_m^* & \alpha_m^{*2} & \cdots & \alpha_m^{*s}\\
0 & 0 & 0 & \cdots & 0 & 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^s\\
0 & 0 & 0 & \cdots & 0 & 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^s\\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & & \vdots\\
0 & 0 & 0 & \cdots & 0 & 1 & \beta_k & \beta_k^2 & \cdots & \beta_k^s
\end{bmatrix}$$

and B_n is the matrix we get by deleting the first row and last column from A_n.

At this point we note that as $n\to\infty$ we have $r\to\infty$ and thus $\alpha_i^{r+j}\to 0$ for fixed i and j. Of course $(\bar{\alpha}_i)^{r+j}\to 0$ as well.

In fact, $\det B_n$ approaches $(-1)^{s(s-1)}$ time a product of Vandermonde determinants and we find

$$\lim_{n\to\infty}\det B_n=\xi(\alpha_1,\alpha_2,\ldots,\alpha_m,\beta_1^*,\beta_2^*,\ldots,\beta_k^*)$$
$$\times\,\xi(\alpha_1^*,\alpha_2^*,\ldots,\alpha_m^*,\beta_1,\beta_2,\ldots,\beta_k),$$

where ξ is the difference product, or a Vandermonde determinant defined as usual by

$$\xi(x_1,x_2,\ldots,x_n)=\prod_{1\leqslant i<j\leqslant n}(x_i-x_j).$$

Now let $D_0^{(n)}(z)$ denote the matrix we get from replacing the first row of $A_n(z)$ by

$$(0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad z \quad z^2 \quad \cdots \quad z^s)$$

and let $D_j^{(n)}(z)$ denote the matrix we get from replacing the first $(j+1)$ rows of $A_n(z)$ by

$$\begin{pmatrix} 1 & z & z^2 & \cdots & z^{s-1} & 0 & 0 & 0 & \cdots & 0 \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{s-1} & 0 & 0 & 0 & \cdots & 0 \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{s-1} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_{j-1} & \alpha_{j-1}^2 & \cdots & \alpha_{j-1}^{s-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \alpha_j & \alpha_j^2 & \cdots & \alpha_j^s \end{pmatrix}$$

so that by repeatedly decomposing $\det(A_n(z))$ we may write

$$\det A_n(z) = z^r \det D_0^{(n)}(z) + \alpha_1^r \det D_1^{(n)}(z) + \alpha_2^r \det D_2^{(n)}(z) + \cdots + \alpha_m^r \det D_m^{(n)}(z).$$

The sum stops at m because the remaining determinants vanish by Laplace's theorem. This decomposition means that

$$cG_{2m}(z)\,\Psi_n(z) = z^{n+m-k} p_0^{(n)}(z) + \alpha_1^{n+m-k} p_1^{(n)}(z) + \alpha_2^{n+m-k} p_2^{(n)}(z) + \cdots + \alpha_m^{n+m-k} p_m^{(n)}(z), \tag{4}$$

where $p_i^{(n)}(z) = \det D_i^{(n)}(z)/\det B_n$.

This is a nice form for $\Psi_n(z)$ as $\lim_{n\to\infty} D_i^{(n)}(z)$ can easily be evaluated in exactly the same manner we evaluated $\lim_{n\to\infty} B_n$. Hence, letting $\sigma = s(s+1)$,

$$\begin{aligned} &\lim_{n\to\infty} p_0^{(n)}(z) \\ &= (-1)^\sigma \frac{\xi(\alpha_1, \alpha_2, ..., \alpha_m, \beta_1^*, \beta_2^*, ..., \beta_k^*)\,\xi(z, \alpha_1^*, \alpha_2^*, ..., \alpha_m^*, \beta_1, \beta_2, ..., \beta_k)}{\xi(\alpha_1, \alpha_2, ..., \alpha_m, \beta_1^*, \beta_2^*, ..., \beta_k^*)\,\xi(\alpha_1^*, \alpha_2^*, ..., \alpha_m^*, \beta_1, \beta_2, ..., \beta_k)} \\ &= \prod_{i=1}^m (z-\alpha_i^*) \prod_{j=1}^k (z-\beta_j). \end{aligned} \tag{5}$$

For $h \geqslant 1$ we find

$$\lim_{n\to\infty} p_h^{(n)}(z) = \frac{(-1)^{\sigma+h}}{(-1)^{h-1}} \frac{\xi(z, \alpha_1, \alpha_2, ..., \alpha_{h-1}, \alpha_{h+1}, ..., \alpha_m, \beta_1^*, \beta_2^*, ..., \beta_k^*)}{\xi(\alpha_h, \alpha_1, \alpha_2, ..., \alpha_{h-1}, \alpha_{h+1}, ..., \alpha_m, \beta_1^*, \beta_2^*, ..., \beta_k^*)} \times \frac{\xi(\alpha_h, \alpha_1^*, \alpha_2^*, ..., \alpha_m^*, \beta_1, \beta_2, ..., \beta_k)}{\xi(\alpha_1^*, \alpha_2^*, ..., \alpha_m^*, \beta_1, \beta_2, ..., \beta_k)}.$$

Thus

$$\lim_{n\to\infty} p_h^{(n)}(z) = (-1)\prod_{i=1}^{h-1}\left(\frac{z-\alpha_i}{\alpha_h-\alpha_i}\right)\prod_{i=h+1}^{m}\left(\frac{z-\alpha_i}{\alpha_h-\alpha_i}\right)$$
$$\times\prod_{j=1}^{k}\left(\frac{z-\beta_j^*}{\alpha_h-\beta_j^*}\right)\prod_{i=1}^{m}(\alpha_h-\alpha_i^*)\prod_{j=1}^{k}(\alpha_h-\beta_j). \tag{6}$$

Actually, this last equation is no accident. Let $E_{s-1}^{(n)}(z)$ denote the matrix we get from replacing the first row of $A_n(z)$ by

$$(1 \quad z \quad z^2 \quad \cdots \quad z^{s-1} \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0)$$

Write Eq. (4) in the form

$$cG_{2m}(z)\,\Psi_n(z) = z^{n+m-k}p_0^{(n)}(z) + q_{s-1}^{(n)}(z),$$

where $p_0^{(n)}(z)$ is as before but $q_{s-1}^{(n)}(z) = \det E_{s-1}^{(n)}/\det B_n$.

Note that $q_{s-1}^{(n)}(z)$ is a polynomial of degree $s = m+k-1$ and, furthermore,

$$q_{s-1}^{(n)}(\beta_i^*) = 0, \qquad i = 1, 2, ..., k$$
$$q_{s-1}^{(n)}(\alpha_j) = -\alpha_j^{n+m-k}p_0^{(n)}(\alpha_j), \qquad j = 1, 2, ..., m.$$

These equations, $m+k$ in number, fix $q_{s-1}^{(n)}(z)$ by uniqueness of interpolation. Hence even before taking limits we must have

$$p_h^{(n)}(z) = (-1)\prod_{i=1}^{h-1}\left(\frac{z-\alpha_i}{\alpha_h-\alpha_i}\right)\prod_{i=h+1}^{m}\left(\frac{z-\alpha_i}{\alpha_h-\alpha_i}\right)\prod_{j=1}^{k}\left(\frac{z-\beta_j^*}{\alpha_h-\beta_j^*}\right)p_0^{(n)}(\alpha_h).$$

Equations (4), (5), and (6) determine the limiting behavior of the polynomials $\{\Psi_n(z)\}$ inside and outside the unit circle. For example, if $|z| > \max\{\alpha_1|, |\alpha_2|, ..., |\alpha_m|\}$ it is clear from Eqs. (4) and (5) that

$$cG_{2m}(z)\,\Psi_n(n) \approx z^{n+m-k}\prod_{i=1}^{m}(z-\alpha_i^*)\prod_{j=1}^{k}(z-\beta_j)$$

and thus for such z

$$\Psi_n(z) \approx z^{n+m-k}\prod_{j=1}^{k}(z-\beta_j)\Big/\prod_{i=1}^{m}(z-\alpha_i).$$

For z on or the exterior to the unit circle these results are well known in terms of the Szegö function; see Grenander and Szegö [6, pp. 50–55].

The interesting case is when $|z| < 1$. However, it is clear that if $|\alpha_h|$ is the unique maximum from $\{|z|, |\alpha_1|, |\alpha_2|, ..., |\alpha_m|\}$, assuming such a unique maximum exists, then (4) and (6) give

$$\lim_{n\to\infty} \frac{cG_{2m}(z)\Psi_n(z)}{\alpha_h^{n+m-k}} = (-1)\prod_{i=1}^{h-1}\left(\frac{z-\alpha_i}{\alpha_h-\alpha_i}\right)\prod_{i=h+1}^{m}\left(\frac{z-\alpha_i}{\alpha_h-\alpha_i}\right)\prod_{j=1}^{k}\left(\frac{z-\beta_j^*}{\alpha_h-\beta_j^*}\right)$$
$$\times\prod_{i=1}^{m}(\alpha_h-\alpha_i^*)\prod_{j=1}^{k}(\alpha_h-\beta_j).$$

Thus

$$\Psi_n(z) \approx \alpha_h^{n+m-k}\left(\frac{1}{\alpha_h-z}\right)\prod_{i=1}^{h-1}\left(\frac{1}{\alpha_h-\alpha_i}\right)\prod_{i=h+1}^{m}\left(\frac{1}{\alpha_h-\alpha_i}\right)\prod_{j=1}^{k}\left(\frac{z-\beta_j^*}{\alpha_h-\beta_j^*}\right)$$
$$\times\prod_{i=1}^{m}\left(\frac{\alpha_h-\alpha_i^*}{z-\alpha_i^*}\right)\prod_{j=1}^{k}(\alpha_h-\beta_j)$$

for α_h such that $|\alpha_h| > \max\{|z|, |\alpha_1|, |\alpha_2|, ..., |\alpha_{h-1}|, |\alpha_{h+1}|, ..., |\alpha_m|\}$.

When there is no unique maximum from $\{|z|, |\alpha_1|, |\alpha_2|, ..., |\alpha_m|\}$, Eqs. (4), (5), and (6) yield similar, but more complicated, behavior. The case when the zeros of $G_{2m}(z)$ are not pairwise distinct is currently being investigated.

5. The Polynomials of Askey, Hahn, and Szegö

Askey [2, pp. 806–811] used the Ramanujun ${}_1\psi_1$ sum to prove that the polynomials

$$S_n^a(z) = \sum_{k=0}^{n} \frac{(aq^2;q^2)_k(a;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}(zq^{-1})^k \tag{7}$$

satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_0^{2\pi} S_n^a(e^{i\theta})\,\overline{S_m^a(e^{i\theta})}\left|\frac{(qe^{i\theta};q^2)_\infty}{(aqe^{i\theta};q^2)_\infty}\right|^2 d\theta$$
$$= \frac{q^{-2n}(a^2q^2;q^2)_n(aq^2;q^2)_\infty^2}{(q^2;q^2)_n(q^2;q^2)_\infty(a^2q^2;q^2)_\infty}\,\delta_{m,n}, \tag{8}$$

assuming $-1 < q < 1$ and defining in the standard way

$$(\sigma;q)_0 = 1, \quad (\sigma;q)_n = \prod_{j=1}^{n}(1-\sigma q^{j-1}), \qquad n = 1, 2, ... \text{ or } n = \infty.$$

These polynomials were considered earlier by Hahn [7], see p. 811 in [2], and before that Szegö [14] considered the case when $a=0$.

We can prove the orthogonality of the polynomials (7) using Theorem 1 and the same idea behind Ismail's [8] proof of the ${}_1\psi_1$-summation. Namely, choose appropriate values for the parameter a, and then use analytic continuation.

When $a=q^{2m}$, $m=1, 2, \dots$, the weight function reduces to

$$w(\theta)=(qz;q^2)_m(qz^{-1};q^2)_m, \qquad z=e^{i\theta}, \tag{9}$$

so the zeros of $w(\theta)$ are $\{q^{-(2m-1)}, q^{-(2m-3)}, \dots, q^{-1}, q^{1}, \dots, q^{(2m-3)}, q^{(2m-1)}\}$. All of the zeros are simple: half of them lie inside the unit circle and the other half lie outside.

Applying Theorem 1 with $dv=d\theta$, $\phi_n(z)=z^n$, $\phi_n^*(z)=1$ we find that

$$\psi_n(z)=\frac{\lambda_n}{\xi(z)} \det\begin{bmatrix} 1 & z & \cdots & z^{m-1} & z^{n+m} & \cdots & z^{n+2m} \\ 1 & q & \cdots & q^{m-1} & q^{n+m} & \cdots & q^{n+2m} \\ 1 & q^3 & \cdots & q^{3(m-1)} & q^{3(n+m)} & \cdots & q^{3(n+2m)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & q^{2m-1} & \cdots & q^{(2m-1)(m-1)} & q^{(2m-1)(n+m)} & \cdots & q^{(2m-1)(n+2m)} \\ 1 & q^{-1} & \cdots & q^{-(m-1)} & q^{-(n+m)} & \cdots & q^{-n+2m)} \\ 1 & q^{-3} & \cdots & q^{-3(m-1)} & q^{-3(n+m)} & \cdots & q^{-3(n+2m)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & q^{-(2m-1)} & \cdots & q^{-(2m-1)(m-1)} & q^{-(2m-1)(n+m)} & \cdots & q^{-(2m-1)(n+2m)} \end{bmatrix}, \tag{10}$$

where $\xi(z)$ denotes the Vandermonde determinant, or difference product, on $\{z, q, q^{-1}, \dots, q^{(2m-1)}, q^{-(2m-1)}\}$, and λ_n is a suitable constant.

Now, let h_k denote the complete symmetric function of degree k on $\{z, q, q^{-1}, \dots, q^{(2m-1)}, q^{-(2m-1)}\}$, and let s_k denote the complete symmetric function on $\{q, q^{-1}, \dots, q^{(2m-1)}, q^{-(2m-1)}\}$. We set $h_0=s_0=1$ and $h_{-k}=s_{-k}=0$ for $k>0$.

Note that

$$h_k=zh_{k-1}+s_k \tag{11}$$

for all integers k.

From Muir [13] we see that

$$\psi_n(z)=\lambda_n \det\begin{bmatrix} h_0 & h_1 & \cdots & h_{m-1} & h_{n+m} & h_{n+m+1} & \cdots & h_{n+2m} \\ h_{-1} & h_0 & \cdots & h_{m-2} & h_{n+m-1} & h_{n+m} & \cdots & h_{n+2m-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{-m+1} & h_{-m+2} & \cdots & h_0 & h_{n+1} & h_{n+2} & \cdots & h_{n+m+1} \\ h_{-m} & h_{-m+1} & \cdots & h_{-1} & h_n & h_{n+1} & \cdots & h_{n+m} \\ h_{-m-1} & h_{-m} & \cdots & h_{-2} & h_{n-1} & h_n & \cdots & h_{n+m-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{-2m} & h_{-2m+1} & \cdots & h_{-m-1} & h_{n-m} & h_{n-m+1} & \cdots & h_n \end{bmatrix},$$

Thus

$$\psi_n(z)=\lambda_n \det\begin{pmatrix} h_n & h_{n+1} & h_{n+2} & \cdots & h_{n+m} \\ h_{n-1} & h_n & h_{n+1} & \cdots & h_{n+m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{n-m} & h_{n-m+1} & h_{n-m+2} & \cdots & h_n \end{pmatrix}$$

and using (11) repeatedly we get

$$\psi_n(z)=\lambda_n \det\begin{pmatrix} h_n & s_{n+1} & s_{n+2} & \cdots & s_{n+m} \\ h_{n-1} & s_n & s_{n+1} & \cdots & s_{n+m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{n-m} & s_{n-m+1} & s_{n-m+2} & \cdots & s_n \end{pmatrix}. \tag{12}$$

Now let

$$A_k=\det\begin{pmatrix} s_{n-k} & s_{n+1} & s_{n+2} & \cdots & s_{n+m} \\ s_{n-(k+1)} & s_n & s_{n+1} & \cdots & s_{n+m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ s_{n-(k+m)} & s_{n-m+1} & s_{n-m+2} & \cdots & s_n \end{pmatrix}$$

and

$$B_k=\det\begin{pmatrix} h_{n-k} & s_{n+1} & s_{n+2} & \cdots & s_{n+m} \\ h_{n-(k+1)} & s_n & s_{n+1} & \cdots & s_{n+m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{n-(k+m)} & s_{n-m+1} & s_{n-m+2} & \cdots & s_n \end{pmatrix},$$

We have $\psi_n/\lambda_n = B_0$ but moreover, from (11) we know that $B_k = zB_{k+1} + A_k$ and that $B_k = 0$ if $k > n$. Therefore,

$$\frac{\psi_n(z)}{\lambda_n} = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0.$$

The problem now is to evaluate A_k in general. We have $2m$ zeros in our weight function but A_k is only a $(m+1)$ by $(m+1)$ determinant. We "fill out" A_k and use Muir [13] in reverse.

That is,

$$A_k = \det\begin{pmatrix} s_0 & s_1 & \cdots & s_{m-2} & s_{n-k+m-1} & s_{n+m} & s_{n+m+1} & \cdots & s_{n+2m-1} \\ s_{-1} & s_0 & \cdots & s_{m-3} & s_{n-k+m-2} & s_{n+m-1} & s_{n+m} & \cdots & s_{n+2m-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ s_{-m+2} & s_{-m+3} & \cdots & s_0 & s_{n-k+1} & s_{n+2} & s_{n+3} & \cdots & s_{n+m+1} \\ s_{-m+1} & s_{-m+2} & \cdots & s_{-1} & s_{n-k} & s_{n+1} & s_{n+2} & \cdots & s_{n+m} \\ s_{-m} & s_{-m+1} & \cdots & s_{-2} & s_{n-k-1} & s_n & s_{n+1} & \cdots & s_{n+m-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ s_{-2m+1} & s_{-2m+2} & \cdots & s_{-m-1} & s_{n-k-m} & s_{n-m+1} & s_{n-m+2} & \cdots & s_n \end{pmatrix}$$

and hence

$$\xi A_k = \det\begin{pmatrix} \text{row}_1 \\ \text{row}_2 \\ \vdots \\ \text{row}_{2m} \end{pmatrix}, \tag{13}$$

where ξ denotes the Vandermonde determinant on $\{q, q^{-1}, ..., q^{(2m-1)}, q^{-(2m-1)}\}$, and

$$\begin{aligned} \text{row}_{2h-1} = [1, q^{2h-1}, ..., q^{(2h-1)(m-2)}, q^{(2h-1)(n-k+m-1)}, q^{(2h-1)(n+m)}, \\ q^{(2h-1)(n+m+1)}, ..., q^{(2h-1)(n+2m-2)}, q^{(2h-1)(n+2m-1)}] \\ \text{row}_{2h} = [1, q^{1-2h}, ..., q^{(1-2h)(m-2)}, q^{(1-2h)(n-k+m-1)}, q^{(1-2h)(n+m)}, \\ q^{(1-2h)(n+m+1)}, ..., q^{(1-2h)(n+2m-2)}, q^{(1-2h)(n+2m-1)}]. \end{aligned}$$

Here is where the spacing of the zeros comes into play, for the determinant in (13) is actually a power of q times a Vandermonde determinant. That is, if we set

$$\begin{aligned} r = -(2m-1)[0 + 1 + \cdots + (m-2) + (n-k+m-1) \\ + (n+m) + (n+m+1) + \cdots + (n+2m-1)], \end{aligned}$$

then

$$A_k = q^r \frac{\left[\begin{array}{c}\xi(1, q^2, q^4, ..., q^{2(m-2)}, q^{2(n-k+m-1)}, \\ q^{2(n+m)}, q^{2(n+m+1)}, ..., q^{2(n+2m-1)})\end{array}\right]}{\xi(q, q^{-1}, q^3, q^{-3}, ..., q^{2m-1}, q^{1-2m})}.$$

As the Christoffel theorem(s) only give the polynomials up to constant multiple factors, we are interested in the ratios of coefficients more than in the coefficients themselves. Here in the coefficient ratio we have a tremendous amount of cancellation. More precisely, we find after some computation that

$$\frac{A_{k+1}}{A_k} = q^{-1} \frac{(1-q^{2(n-k)})(1-q^{2(k+m+1)})}{(1-q^{2(n-k+m-1)})(1-q^{2(k+1)})}.$$

However, this is easily seen to be

$$\frac{A_{k+1}}{A_k} = q^{-1}\left(\left[\frac{(q^2;q^2)_{n-k}}{(q^2;q^2)_{n-k-1}}\right]\left[\frac{(q^{2m}q^2;q^2)_{k+1}}{(q^{2m}q^2;q^2)_k}\right]\right) \Bigg/ \left(\left[\frac{(q^{2m};q^2)_{n-k}}{(q^{2m};q^2)_{n-k-1}}\right]\left[\frac{(q^2;q^2)_{k+1}}{(q^2;q^2)_k}\right]\right)$$

and thus

$$\frac{A_{k+1}}{A_k} = q^{-1}\left(\frac{(aq^2;q^2)_{k+1}(a;q^2)_{n-(k+1)}}{(q^2;q^2)_{k+1}(q^2;q^2)_{n-(k+1)}}\right) \Bigg/ \left(\frac{(aq^2;q^2)_k(a;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}\right),$$

where $a = q^{2m}$.

At this point we know the $S_n^a(z)$ for this choice of a are orthogonal on the unit circle. However, we still need the right hand side of Eq. (8). If we consider the monic polynomials

$$\Phi_n(z) = \frac{(q^2;q^2)_n q^n}{(aq^2;q^2)_n} S_n^a(z)$$

then from Geronimus [4] we have

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi_n(z)\overline{\Phi_n(z)}w(\theta)\,d\theta = c_0 \prod_{k=1}^{n} (1-|\Phi_k(0)|^2), \tag{14}$$

but

$$\Phi_k(0) = \frac{A_0}{A_k} = \frac{(a;q^2)_k}{(aq^2;q^2)_k} q^k = \frac{(1-a)}{(1-aq^{2k})} q^k,$$

so

$$(1-|\Phi_k(0)|^2)=\left[1+\frac{1-a}{1-aq^{2k}}q^k\right]\left[1-\frac{1-a}{1-aq^{2k}}q^k\right]$$
$$=\left[\frac{(1+q^k)(1-aq^k)}{(1-aq^{2k})}\right]\left[\frac{(1-q^k)(1+aq^k)}{(1-aq^{2k})}\right]$$
$$=\frac{(1-q^{2k})(1-a^2q^{2k})}{(1-aq^{2k})^2}.$$

Substituting these into (14) we have

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi_n(z)\overline{\Phi_n(z)}w(\theta)\,d\theta=c_0\frac{(q^2;q^2)_n(a^2q^2;q^2)_n}{(aq^2;q^2)_n^2} \tag{15}$$

which, multiplying through by

$$\left[\frac{(aq^2;q^2)_n}{(q^2;q^2)_n}q^{-n}\right]^2$$

gives

$$\frac{1}{2\pi}\int_0^{2\pi}S_n^a(e^{i\theta})\overline{S_m^a(e^{i\theta})}w(\theta)\,d\theta=c_0q^{-2n}\frac{(a^2q^2;q^2)_n}{(q^2;q^2)_n}. \tag{16}$$

Comparing Eqs. (8) and (16) we must have (if $a=q^{2m}$)

$$c_0=\frac{(aq^2;q^2)_\infty^2}{(q^2;q^2)_\infty(a^2q^2;q^2)_\infty}.$$

This is cheating. After all, we are trying to derive (8) a different way. In fact, we can, for the spacing of the zeros of $w(\theta)$ allows us to compute all of the moments, including c_0, directly. We write

$$w(\theta)=\frac{z^{-m}(-1)^mq^{m^2}}{\xi}\det\begin{pmatrix}1 & z & z^2 & \cdots & z^{2m}\\ 1 & q & q^2 & \cdots & q^{2m}\\ 1 & q^{-1} & q^{-2} & \cdots & q^{-2m}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & q^{2m-1} & q^{2(2m-1)} & \cdots & q^{(2m)(2m-1)}\\ 1 & q^{(1-2m)} & q^{2(1-2m)} & \cdots & q^{(2m)(1-2m)}\end{pmatrix},$$

where ξ denotes the Vandermonde determinant, or difference product, on $\{q, q^{-1}, ..., q^{(2m-1)}, q^{-(2m-1)}\}$. Therefore

$$c_{n-m} = \frac{q^{m^2}(-1)^{m+n}}{\xi} \det \begin{pmatrix} \mathrm{row}_1 \\ \mathrm{row}_2 \\ \vdots \\ \mathrm{row}_{2m} \end{pmatrix},$$

where

$$\begin{aligned} \mathrm{row}_{2h-1} &= [1, q^{2h-1}, ..., q^{(2h-1)(n-1)}, \\ &\qquad q^{(2h-1)(n+1)}, q^{(2h-1)(n+2)}, ..., q^{(2h-1)(2m)}] \\ \mathrm{row}_{2h} &= [1, q^{1-2h}, ..., q^{(1-2h)(n-1)}, \\ &\qquad q^{(1-2h)(n+1)}, q^{(1-2h)(n+2)}, ..., q^{(1-2h)(2m)}]. \end{aligned}$$

We get

$$\begin{aligned} c_{n-m} &= q^{m^2+(n-2m)(2m-1)}(-1)^{m+n} \\ &\quad \times \frac{\xi(1, q^2, q^4, ..., q^{2(n-1)}, q^{2(n+1)}, q^{2(n+2)}, ..., q^{4m})}{\xi(1, q^2, q^4, ..., q^{4m-2})} \\ &= q^{m^2+(n-2m)(2m-1)}(-1)^{m-1} \\ &\quad \times \frac{\xi(1, q^2, q^4, ..., q^{2(n-1)}, q^{2(n+1)}, q^{2(n+2)}, ..., q^{4m-2}, q^{4m})}{\xi(1, q^2, q^4, ..., q^{2(n-1)}, q^{2(n+1)}, q^{2(n+2)}, ..., q^{4m-2}, q^{2n})} \\ &= q^{m^2+(n-2m)(2m-1)}(-1)^{m-1} \frac{[(1-q^{4m})(q^2-q^{4m}) \cdots (q^{2n-2}-q^{4m})]}{[(1-q^{2n})(q^2-q^{2n}) \cdots (q^{2n-2}-q^{2n})]} \\ &\quad \times \frac{[(q^{2n+2}-q^{4m})(q^{2n+4}-q^{4m}) \cdots (q^{4m-2}-q^{4m})]}{[(q^{2n+2}-q^{2n})(q^{2n+4}-q^{2n}) \cdots (q^{4m-2}-q^{2n})]} \\ &= q^{(n-m)^2}(-1)^{n-m} \frac{(1-q^{4m})(1-q^{4m-2}) \cdots (1-q^{4m-2n+2})}{(1-q^{2n})(1-q^{2n-2}) \cdots (1-q^2)}. \end{aligned}$$

Thus

$$\begin{aligned} c_k &= q^{k^2}(-1)^k \frac{(1-q^{2m-2k+2})(1-q^{2m-2k+4}) \cdots (1-q^{4m})}{(1-q^2)(1-q^4) \cdots (1-q^{2m+2k})} \\ &= q^{k^2}(-1)^k \frac{(q^{2m+2-2k}; q^2)_\infty / (q^{4m+2}; q^2)_\infty}{(q^2; q^2)_\infty / (q^{2m+2k+2}; q^2)_\infty} \\ &= q^{k^2}(-1)^k \frac{(aq^{2-2k}; q^2)_\infty (aq^{2+2k}; q^2)_\infty}{(q^2; q^2)_\infty (a^2q^2; q^2)_\infty}. \end{aligned}$$

In particular,

$$c_0 = \frac{(aq^2; q^2)_\infty^2}{(q^2; q^2)_\infty (a^2q^2; q^2)_\infty}$$

for $a = q^{2m}$ and we conclude that (8) is precisely true for these a and, by analytic continuation, for all a with $a^2q^2 < 1$. Moreover,

$$\frac{c_k}{c_0} = q^{k^2}(-1)^k \frac{(aq^{2-2k}; q^2)_\infty (aq^{2+2k}; q^2)_\infty}{(aq^2; q^2)_\infty^2} = \frac{(a^{-1}; q^2)_k}{(aq^2; q^2)_k} (aq)^k$$

implies

$$\sum_{-\infty}^{\infty} \frac{(a^{-1}; q^2)_k}{(aq^2; q^2)_k} (aqe^{i\theta})^k = \frac{(qe^{i\theta}; q^2)_\infty (qe^{-i\theta}; q^2)_\infty (q^2; q^2)_\infty (a^2q^2; q^2)_\infty}{(aqe^{i\theta}; q^2)_\infty (aqe^{-i\theta}; q^2)_\infty (aq^2; q^2)_\infty^2}. \tag{17}$$

This instance of the ${}_1\psi_1$-sum was the basic ingredient in Askey's proof of the orthogonality relation for the polyomials $\{S_n^a(z)\}$. Our approach gives (17) as a by-product.

Note that one can also evaluate c_0 in (16) by setting $m = n = 0$ in (16) and expand the weight function using the q-binomial theorem.

6. Remarks

Alfaro and Marcellan [1] considered the case when a weight $dv(\theta)$ on the unit circle is to be multiplied by $|z-\beta|^2$, $z = e^{i\theta}$. Later, Godoy and Marcellan [5] obtained a different version of our Theorem 1. Their version, however, is a determinantal expression involving both the ϕ_n's and their kernel polynomials. We believe that our determinantal representation is much more convenient. Neither [1] nor [5] contain any application requiring the full use of the general determinant representation.

The expression (10) is curious because it provides a representation for a multiple of $S_n^a(z)$ as a polynomial in the two variables z and q. As orthogonal polynomials the dependence on z is important but in many combinatorial problems an expression of the polynomials under consideration in z (or x on the line) and q is very useful. See Viennot [18] and Ismail, Stanton, and Viennot [9].

The work [3] surveys mostly papers characterizing Appell and q-Appell polynomials on the unit circle. Kholodov [10] solved the more general case of characterizing Sheffer sequences which are also orthogonal on the unit circle.

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REFERENCES

1. M. ALFARO AND F. MARCELLAN, Recent trends in orthogonal polynomials on the unit circle, *in* "Orthogonal Polynomials and Their Applications" (C. Brezinski et al., Eds.), pp. 3–14, Baltzer, Basel, 1991.
2. R. ASKEY, Comments on paper 1926-6, *in* "Collected Papers of G. Szegö," Vol. 1, pp. 806–811, Birkhäuser, Boston, 1982.
3. P. GARCIA-LAZARS, F. MARCELLAN, AND C. TASIS, On a Szegö's result: Generating sequences of orthogonal polynomials on the unit circle, *in* "Orthogonal Polynomials and Their Applications" (C. Brezinski et al., Eds.), pp. 271–274, Baltzer, Basel, 1991.
4. YA. L. GERONIMUS, "Polynomials Orthogonal on a Circle and Interval," pp. 132–133, Pergamon, Elmsford, NY, 1960.
5. E. GODOY AND F. MARCELLAN, An analog of the Christoffel formula for polynomial modification of a measure on the unit circle, *Boll. Un. Mat. Ital. A (7)* **5** (1991), 1–12.
6. U. GRENANDER AND G. SZEGÖ, "Toeplitz Forms and Their Applications," Univ. of California Press, Berkeley/Los Angeles, 1958; reprinted by Chelsea, New York, 1984.
7. W. HAHN, Über Polynome, die gleichzeitig zwei verschiedenen Orthogonalsystemen angehören, *Math. Nach.* **2** (1949), 263–278.
8. M. E. H. ISMAIL, A simple proof of Ramanujan's ${}_1\psi_1$ sum, *Proc. Amer. Math. Soc.* **63** (1977), 185–186.
9. M. E. H. ISMAIL, D. STANTON, AND G. VIENNOT, The combinatorics of q-Hermite polynomials and the Askey–Wilson integral, *European J. Combin.* **8** (1987), 379–392.
10. A. N. KHOLODOV, The umbral calculus and orthogonal polynomials, *Acta Appl. Math.* **19** (1990), 1–54.
11. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Strong and weak convergence of orthogonal polynomials, *Amer. J. Math.* **109** (1987), 239–282.
12. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Extensions of Szegö's theory of orthogonal polynomials, III, *Constr. Approx.* **3** (1987), 73–96.
13. T. MUIR, "A Treatise on the Theory of Determinants," pp. 334–335, Dover, New York, 1960.
14. G. SZEGÖ, Ein Beitrag zur Theorie der Thetafunktionen, *Sitzungsber. Berlin. Akad.* (1926), 242–252; reprinted *in* "Collected Papers of G. Szegö," Vol. 1, pp. 795–805, Birkhäuser, Boston, 1982.
15. G. SZEGÖ, "Orthogonal Polynomials," 4th. ed., Amer. Math. Soc. Providence, RI, 1975.
16. V. B. UVAROV, Relation between polynomials orthogonal with different weights *Dokl. Akad. Nauk USSR* **126** (1959), 33–36. [In Russian]
17. V. B. UVAROV, The connection between systems of polynomials that are orthogonal with respect to different distribution functions, *USSR Comput. Math. and Math. Phys.* **9** (1969), 25–36.
18. G. VIENNOT, "Une théorie combinatoire des polynômes orthogonaux généraux," Lecture Notes, Université du Quebec à Montréal, Montréal, 1983.